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THESIS

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OF AIRPLANE SPIN EQUATIONS
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by

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by

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ABSTRACT

The general equations of motion for a rigid body are derived in cylindrical coordinates by Lagrangian dynamics and used to model the motion of an airplane in a steady spin. After simplification, the equations are cast into a form utilizing conventional aerodynamic derivatives along with other derivatives which may be significant in spins. An iterative numerical solution procedure is outlined which should simplify the problem of solving the nonlinear differential equations, and relationships between the Euler Angles used in the equations and the more familiar ordered set of pitch, roll, and yaw are derived to permit computer input and output of orientation to be more easily visualized. The inverse problem is also considered, and equations are derived to translate spin test data into the six Lagrange coordinates used so that computer calculations may be checked against test results and actual aerodynamic forces can be compared with computed values. The paper concludes with a discussion of the implications of this spin analysis along with applications and extensions.

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TABLE OF SYMBOLS

The definitions of the symbols used in this paper are as follows:

cg	Center of gravity
$\bar{c}$	Mean aerodynamic chord, feet
$C_{F_X}, C_{F_Y}, C_{F_Z}$	Aerodynamic force coefficient along the subscripted axis, dimensionless
$C_{M_X}, C_{M_Y}, C_{M_Z}$	Aerodynamic moment coefficient about the subscripted axis, dimensionless
F_X, F_Y, F_Z	Aerodynamic force along the subscripted axis, lbs.
F_{q_r}	Lagrange generalized non-conservative force, lbs.
F_{z_0}, F_R, F_γ F_θ, F_ϕ, F_ψ	Lagrange generalized force in the subscripted coordinates, lb. or ft - lbs.
g	Acceleration of gravity, ft/sec^2
g_X, g_Y, g_Z	Airplane cg acceleration components along the subscripted axis, ft/sec^2
I_X, I_Y, I_Z	Airplane principal moments of inertia, slug - ft^2
M	Airplane mass, slugs
M_X, M_Y, M_Z	Moments about the subscripted axis, ft - lbs.
n	Sequential subscript, integer
p, q, r	Angular rates of rotation about the airplane X, Y, and Z principal axes, respectively
q_r	Lagrange generalized coordinate
R	Spin radius, feet
S	Airplane wing area, ft^2
T	Lagrange kinetic energy term, ft - lbs.
t	Time, seconds

Δt	Increment of time, seconds
V	Lagrange potential energy term, ft - lbs.
V_{cg}	Airplane cg velocity ft/sec
$V_{RWx}, V_{RWy}, V_{RWz}$	Airplane cg relative wind velocity component along the subscripted axis, ft/sec
X, Y, Z	Airplane principal axis system
X_1, Y_1, Z_1	Reference Cartesian axis system
Z_0	Airplane altitude, feet
$\dot{Z}_0$	Rate of ascent, ft/sec
α	Airplane angle of attack relative to principal axis system, degrees
α_{ij}	Direction cosines in terms of Euler angles (see Table I)
β	Airplane yaw angle, degrees
γ	Lagrange generalized coordinate indicating direction of spin radius vector, radians
$\dot{\gamma}$	Spin rate, radians/sec
$\delta_a, \delta_r, \delta_h$	Aileron, rudder and elevator deflections, degrees
θ, ψ, ϕ	Airplane principal axis system orientation Euler angles with respect to the X_1, Y_1, Z_1 system, radians
θ_P, ϕ_R, ψ_Y	Airplane orientation in pitch, roll, and yaw with respect to the X_1, Y_1, Z_1 system, radians
$\dot{\theta}_P, \dot{\phi}_R, \dot{\psi}_Y$	Pitch, roll and yaw rates, radians/sec
$\varepsilon_{X_1}, \varepsilon_{Y_1}, \varepsilon_{Z_1}$	Incremental angular rotations about the X_1, Y_1, Z_1 system axes, radians
ρ	Air density, slugs/ft^3
$\omega_x, \omega_y, \omega_z$	Angular rates of rotation about the airplane principal axes, equivalent to p, q, and r, radians/sec

I. INTRODUCTION

A. FOREWORD AND ACKNOWLEDGEMENTS

The intent of this work was to develop an alternative to the conventional method of computing spin characteristics which thus far has not been uniformly successful in predicting airplane spin modes. In this effort, the author has tried to approach the problem with the capabilities of modern computers in mind and devise a method of solving directly for spin modes rather than mathematically tracking airplane motion through all intervening phases of motion preceding the spin. If any success is realized through this approach as outlined herein, it will be due in no small measure to the tireless efforts of Lieutenant R. L. Champoux, USN, whose contribution of months of hard work and extensive knowledge of computer programming were instrumental in the evolution of this approach. Lieutenant Champoux's separate work on the computer aspects of this analysis may be found in Ref. 9. The author is also indebted to Professors L. V. Schmidt, T. H. Gawain, and M. H. Redlin, of the U. S. Naval Postgraduate School whose counsel helped shape many aspects of this work.

B. BACKGROUND

Airplane spin or autogyration is a mode of motion exhibited by virtually all fixed wing aircraft. It is at best an undesirable flight maneuver and, with growing

frequency in new airplane designs, a terminal maneuver from which recovery cannot be effected. Although the flight conditions which lead to spins can be avoided, pilots of military light attack airplanes and especially fighter airplanes do not have that option if pressed into flying to the "edge of the airplane's flight envelope" by the mission or a tactical situation.

The need to utilize the edge of the flight envelope in a combat role is readily apparent when one considers that it is only in this fringe area that the superiority of one airplane-pilot combination over another can be realized. However, less obvious is the extent of additional exposure necessary for training in this area if a pilot is to become proficient in a particular airplane under such conditions. Thus, flight near the edge of the envelope must be more than an occasional event for the military pilot. Unfortunately, loss rates due to irrecoverable spin modes in advanced tactical jet airplanes frequently makes effective training in this area economically prohibitive. Thus the margin of superiority built into the airplane at great expense is in many cases lost.

It is in recognition of this fact, and the fact that spin predictions based on conventional analysis techniques have not shown much success, that this effort was undertaken. It is not expected that this analysis of airplane spin will lead directly to solutions of the spin problem itself, but it is hoped that this work will cast the problem in a new mathematical perspective, from which additional insight into the

spin problem might be gained towards a goal of eventually designing out irrecoverable spin modes.

C. GENERAL

Historically airplane spin studies have employed forms of the same equations of motion which are used in the small perturbation approach to airplane dynamics. This method of mathematically modeling the spin appeared promising because the equations could be readily integrated by numerical methods to produce time-histories of characteristic parameters which could then be compared directly with flight test time-histories. It was also apparent that the body axis coordinate system used in the classical approach provided simpler equations for random tumbling motion than those obtained using an inertial system of coordinates. Thus the post-stall and incipient spin gyrations of the airplane, which in many cases are of equal concern, can be more readily tracked by a computer in a body axis system than an inertial system. The degree of success realized by thus extending the classical approach however depends to a great extent on the judgment and intuition of the investigator in leaving just enough coupling in the equations to permit satisfactory representation of the motion without introducing excessive non-linearities. Frequently this process degenerated to a trial and error parametric approach without much insight being gained and without general validity being obtained for all airplane configurations or spin modes.

On the other hand, the treatment of the spin problem in an inertial coordinate system appears to add undue complexity to the computer tracking problem, and indeed for random tumbling motion the non-linearities become excessive. However, if the steady, or quasi-steady[1] spin is to be studied, an inertial cylindrical system offers a number of distinct advantages, among which are the ability to differentiate between motion due to spinning and motion due to tumbling[2], and a sufficient reduction in the complexity of the coupling to permit computer solution of the exact equations[3]. Since the intent of this study was to develop a generalized method of solving for steady spin modes, the equations of motion were derived in an inertial coordinate system. The details of this coordinate system are shown in Figure 1, and discussed in some detail in Section III.

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[1]Spinning motion in which state variables have a mean component and an oscillatory component.

[2]Random motion inconsistent with steady or quasi-steady spin motion.

[3]No small quantity approximations are made in either the derivation or simplification of the equations of motion.

II. SCOPE

This work was preceded by a survey of current airplane spin literature [Ref. 1] to determine the present state-of-the-art in airplane spin analysis. From that study it was evident that possibly another avenue of approach to the problem ought to be explored. Thus, rather than attacking the spin problem on the broad formidable front of mathematically following an airplane through all transient gyrations from stall to spin, it was decided to limit the scope of the investigation and probe directly into the final steady state motion of the spin where a force and moment equilibrium situation might yield to simpler mathematical representation. In this approach no restrictions were imposed on the form of the airplane aerodynamic data to be used in this analysis other than requiring that it be referenced to the airplane's center of gravity and principal axes. By leaving the details of the makeup of such data somewhat general, the analysis is freed from any incumberances of classical aerodynamic derivatives and thus can proceed more directly to the derivation of the general equations of motion. Although conventional force and moment derivatives are suggested for use in initial computer programming, subsequent work may suggest a more appropriate form of aerodynamic force and moment data, or possibly more representative aerodynamic derivatives may be devised.

III. DERIVATION OF THE EQUATIONS OF MOTION

The coordinate system used in the derivation of the equations of motion is a combination of an inertial cylindrical system and the airplane principal axis system. The vertical (z) axis of the cylindrical system corresponds to the central axis of the motion or "spin axis". The advantage of the cylindrical system is that it locates the airplane center of gravity in convenient terms of an altitude coordiante (z_o), spin radius coordinate (R) and angular position coordinate (γ), when the airplane is in a steady-state spin. The orientation of the airplane about its center of gravity with respect to the inertial cylindrical system is specified in terms of Euler angles. A cartesian coordinate system is fixed at the cg position on the (R) vector having its (x_1) axis in the (-R) direction and its (z_1) axis parallel to and in the opposite direction of (z_o). This cartesian system provides a reference for the orientation Euler angles. Thus for zero values of the Euler Angles (θ), (ψ), and (ϕ), the airplane would be upright, wings level, and have the (X) axis of its principal axis system pointed inward, along the (R) vector, directly at the spin axis. The details of this hybrid coordinate system are shown in Fig. 1 along with the positive directions of the position coordinates. A general schematic of Euler angle relationships is contained in Fig. 2 , and equations for the direction cosines of the airplane axes expressed in terms of

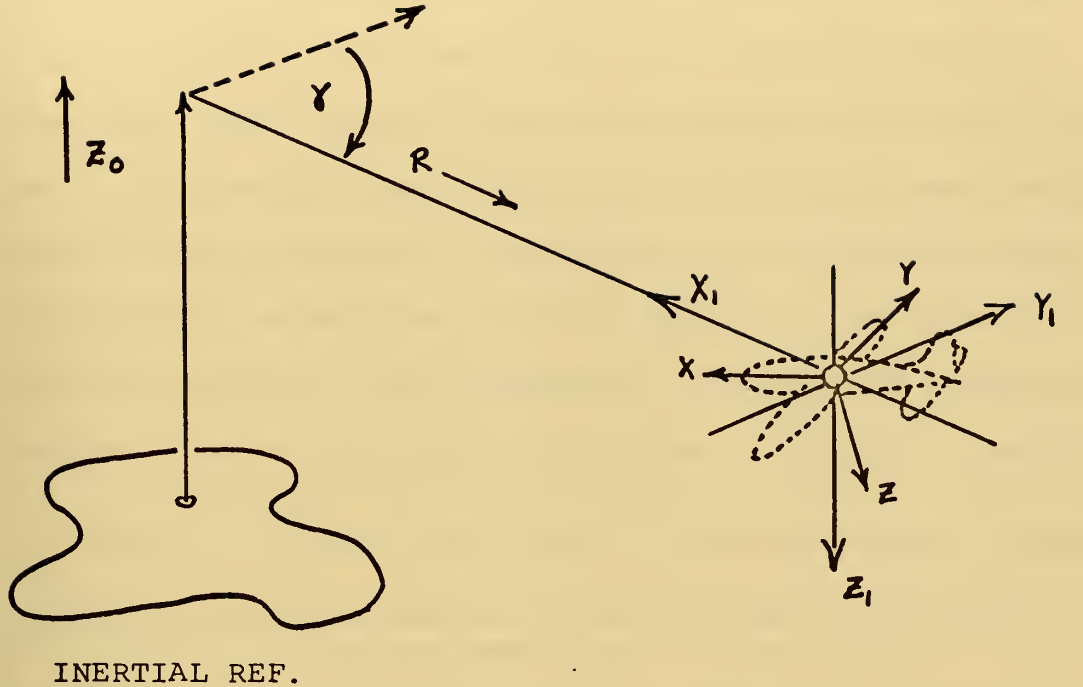

Fig. 1. The Coordinate System Used to Model the Spin

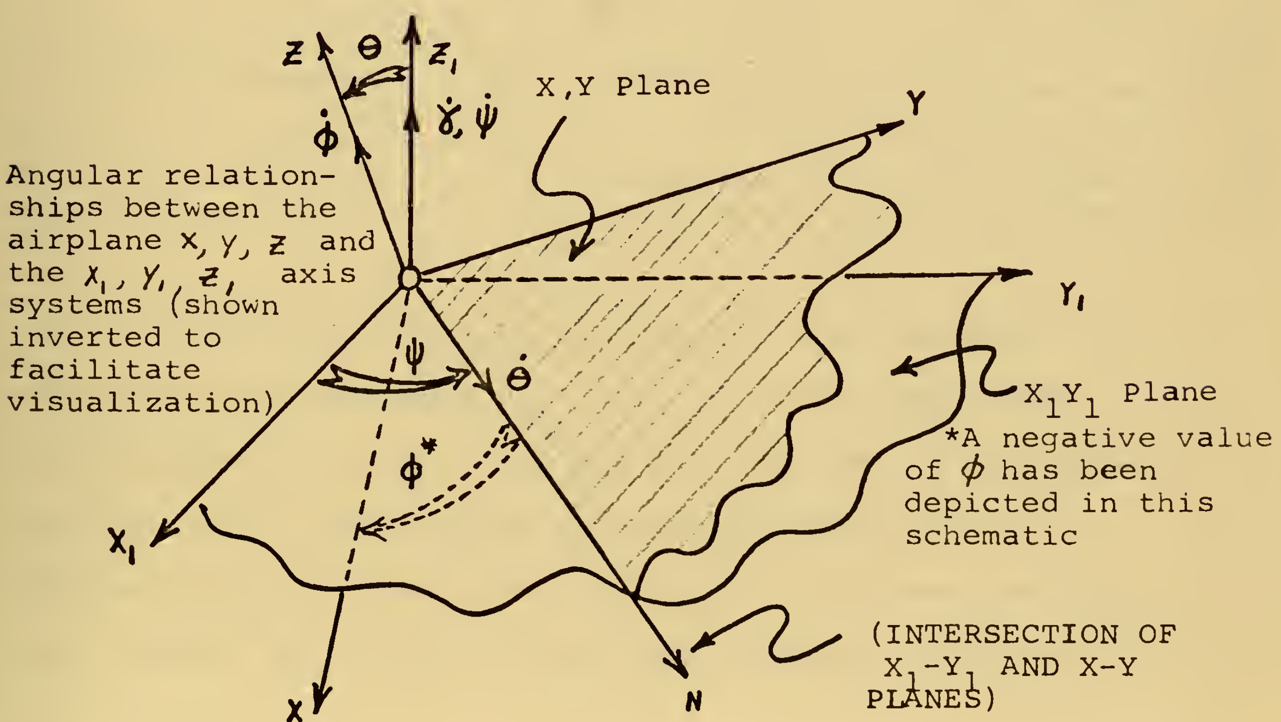

Fig. 2. A General Schematic of Euler Angle Relationships

Euler angles are listed in table I. The rationale behind this choice of coordinate systems is that it more simply (in a mathematical sense) represents the motion being modeled. Thus even for oscillatory spins the motion would be readily recognizable and translatable into mean values of the six coordinates (three position and three orientation). In addition, since the principal objective is to solve for steady spin modes, the complexity of stall and departure motion in this coordinate system has no impact on the results.

Table I: Direction Cosines in Terms of Euler Angles[4]

Cosines of Angles between X, Y, Z and X_1, Y_1, Z_1			
	X	Y	Z
X_1	$\alpha_{11} = \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta$	$\alpha_{21} = -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta$	$\alpha_{31} = \sin\theta\sin\psi$
Y_1	$\alpha_{12} = \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta$	$\alpha_{22} = -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta$	$\alpha_{32} = -\sin\theta\cos\psi$
Z_1	$\alpha_{13} = \sin\theta\sin\phi$	$\alpha_{23} = \sin\theta\cos\phi$	$\alpha_{33} = \cos\theta$

With the coordinate system defined, the equations of motion may be expressed in terms of the six generalized coordinates z_o, R, γ, θ, ψ and ϕ by the use of Lagrangian mechanics. The only simplifying assumption made in the derivation of the equations is that the airplane develops zero thrust in the spin. This is frequently the case in an

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[4]Reference 2, page 158.

actual spin situation due to jet engine stalls and/or flame-outs resulting from inlet airflow distortion. However, the restriction imposed by this assumption can be easily removed by including negative dissipative work terms on the right hand sides of the Lagrange equations which follow. Such terms would include X, Y and Z force components as well as force and moment derivatives about all three axes due to α and β, $\dot{\alpha}$ and $\dot{\beta}$, and the angular rates p, q, and r.

The derivation of the equations of motion is best illustrated by first writing the general Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = F_{q_r} \quad , r = 1, \ldots 6 \tag{1}$$

Where:

T = System kinetic energy

V = System potential energy

q_r = A generalized coordinate, r = 1,2,···,6

Fq_r = A generalized force due to non-conservative forces, r = 1,2,···,6

For this particular coordinate system, T can be written as:

$$T = \tfrac{1}{2}M\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{z}_0)^2\right\} + \tfrac{1}{2}\left\{I_x\omega_x^2 + I_Y\omega_Y^2 + I_z\omega_z^2\right\} \tag{2}$$

Where I_X, I_Y, I_Z, ω_X, ω_Y, ω_Z, are moments of inertia and angular rates of rotation about the airplane's principal axes.

Using Figs. 1 and 2, the angular rates relative to the inertial reference can be written in terms of the spin coordinates as follows:

$$\omega_x = (\dot{\psi}+\dot{\gamma})\alpha_{13} + \dot{\theta}\cos\phi \tag{3}$$

$$\omega_y = (\dot{\psi}+\dot{\gamma})\alpha_{23} - \dot{\theta}\sin\phi \tag{4}$$

$$\omega_z = (\dot{\psi}+\dot{\gamma})\alpha_{33} + \dot{\phi} \tag{5}$$

Note that $\dot{\gamma}$ is included to account for the rotation of the X_1, Y_1, Z_1 system.

The partial derivatives $\partial T/\partial \dot{q}_r$ for $\dot{q}_r = \dot{z}_0, \dot{R}, \dot{\gamma}, \dot{\theta}, \dot{\psi}$ and $\dot{\phi}$ are:

$$\frac{\partial T}{\partial \dot{z}_0} = M\dot{z}_0 \tag{6}$$

$$\frac{\partial T}{\partial \dot{R}} = M\dot{R} \tag{7}$$

$$\frac{\partial T}{\partial \dot{\gamma}} = MR^2\dot{\gamma} + I_x\omega_x\alpha_{13} + I_y\omega_y\alpha_{23} + I_z\omega_z\alpha_{33} \tag{8}$$

$$\frac{\partial T}{\partial \dot{\theta}} = I_x\omega_x\cos\phi - I_y\omega_y\sin\phi \tag{9}$$

$$\frac{\partial T}{\partial \dot{\psi}} = I_x\omega_x\alpha_{13} + I_y\omega_y\alpha_{23} + I_z\omega_z\alpha_{33} \tag{10}$$

$$\frac{\partial T}{\partial \dot{\phi}} = I_z\omega_z \tag{11}$$

And the time derivatives of these are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_o}\right) = M\ddot{z}_o \tag{12}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}}\right) = M\ddot{R} \tag{13}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\gamma}}\right) &= M\left(2R\dot{R}\dot{\gamma} + R^2\ddot{\gamma}\right) + I_x\left(\dot{\omega}_x\alpha_{13} + \omega_x\dot{\alpha}_{13}\right) \\ &\quad + I_y\left(\dot{\omega}_y\alpha_{23} + \omega_y\dot{\alpha}_{23}\right) + I_z\left(\dot{\omega}_z\alpha_{33} + \omega_z\dot{\alpha}_{33}\right)\end{aligned} \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_x\left(\dot{\omega}_x\cos\phi - \dot{\phi}\,\omega_x\sin\phi\right) - I_y\left(\dot{\omega}_y\sin\phi + \dot{\phi}\,\omega_y\cos\phi\right) \tag{15}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) &= I_x\left(\dot{\omega}_x\alpha_{13} + \omega_x\dot{\alpha}_{13}\right) + I_y\left(\dot{\omega}_y\alpha_{23} + \omega_y\dot{\alpha}_{23}\right) \\ &\quad + I_z\left(\dot{\omega}_z\alpha_{33} + \omega_z\dot{\alpha}_{33}\right)\end{aligned} \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = I_z\dot{\omega}_z \tag{17}$$

Next, the partial derivatives $\partial T/\partial q_r$ are obtained:

$$\frac{\partial T}{\partial z_o} = 0 \tag{18}$$

$$\frac{\partial T}{\partial R} = M\dot{\gamma}^2 R \tag{19}$$

$$\frac{\partial T}{\partial \gamma} = 0 \tag{20}$$

$$\frac{\partial T}{\partial \theta} = I_x \omega_x \frac{\partial \omega_x}{\partial \theta} + I_y \omega_y \frac{\partial \omega_y}{\partial \theta} + I_z \omega_z \frac{\partial \omega_z}{\partial \theta} \tag{21}$$

$$\frac{\partial T}{\partial \psi} = I_x \omega_x \frac{\partial \omega_x}{\partial \psi} + I_y \omega_y \frac{\partial \omega_y}{\partial \psi} + I_z \omega_z \frac{\partial \omega_z}{\partial \psi} \tag{22}$$

$$\frac{\partial T}{\partial \phi} = I_x \omega_x \frac{\partial \omega_x}{\partial \phi} + I_y \omega_y \frac{\partial \omega_y}{\partial \phi} + I_z \omega_z \frac{\partial \omega_z}{\partial \phi} \tag{23}$$

And from $V = MgZ_o$ the values of $\partial V/\partial q_r$ are:

$$\frac{\partial V}{\partial z_o} = Mg \tag{24}$$

$$\frac{\partial V}{\partial R} = \frac{\partial V}{\partial \gamma} = \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \psi} = \frac{\partial V}{\partial \phi} = 0 \tag{25}$$

Next, evaluating the quantities $\dot{\omega}_k$, $\dot{\alpha}_{i3}$ and $\partial \alpha_{i3}/\partial q_r$ where $k = x, y$ or z and $i = 1, 2,$ or 3, and $r = 4, 5,$ or 6:

$$\dot{\omega}_x = (\ddot{\psi}+\ddot{\gamma})\alpha_{13} + (\dot{\psi}+\dot{\gamma})\dot{\alpha}_{13} + \ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi \tag{26}$$

$$\dot{\omega}_y = (\ddot{\psi}+\ddot{\gamma})\alpha_{23} + (\dot{\psi}+\dot{\gamma})\dot{\alpha}_{23} - \ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi \tag{27}$$

$$\dot{\omega}_z = (\ddot{\psi}+\ddot{\gamma})\alpha_{33} + (\dot{\psi}+\dot{\gamma})\dot{\alpha}_{33} + \ddot{\phi} \tag{28}$$

$$\dot{\alpha}_{13} = \dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi \tag{29}$$

$$\dot{\alpha}_{23} = \dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi \tag{30}$$

$$\dot{\alpha}_{33} = -\dot{\theta} \sin\theta \tag{31}$$

$$\frac{\partial \alpha_{13}}{\partial \theta} = \cos\theta \sin\phi \tag{32}$$

$$\frac{\partial \alpha_{13}}{\partial \phi} = \sin\theta \cos\phi \tag{33}$$

$$\frac{\partial \alpha_{23}}{\partial \theta} = \cos\theta \cos\phi \tag{34}$$

$$\frac{\partial \alpha_{23}}{\partial \phi} = -\sin\theta \sin\phi \tag{35}$$

$$\frac{\partial \alpha_{33}}{\partial \theta} = -\sin\theta \tag{36}$$

$$\frac{\partial \alpha_{13}}{\partial \psi} = \frac{\partial \alpha_{23}}{\partial \psi} = \frac{\partial \alpha_{33}}{\partial \psi} = \frac{\partial \alpha_{33}}{\partial \phi} = 0 \tag{37}$$

The partial derivatives $\frac{\partial \omega_x}{\partial \theta}$, etc., are also easily obtained, and will not be shown here.

Finally, by substituting the derivatives defined in equations (12) through (25), six Lagrange equations of motion can be written in terms of the quantities defined in equations (26) through (37) and the partial derivatives of the angular velocities are as follows:

$$M\ddot{z}_o + Mg = F_{z_o} \tag{38}$$

$$M\ddot{R} - M\dot{\gamma}^2 R = F_R \tag{39}$$

$$MR(2\dot{R}\dot{\gamma} + R\ddot{\gamma}) + I_x(\dot{\omega}_x \alpha_{13} + \omega_x \dot{\alpha}_{13}) + I_y(\dot{\omega}_y \alpha_{23} + \omega_y \dot{\alpha}_{23})$$
$$+ I_z(\dot{\omega}_z \alpha_{33} + \omega_z \dot{\alpha}_{33}) = F_\gamma \tag{40}$$

$$I_x(\dot{\omega}_x \cos\phi - \dot{\phi}\omega_x \sin\phi) - I_y(\dot{\omega}_y \sin\phi + \omega_y \dot{\phi} \cos\phi)$$
$$-\left(I_x \omega_x \frac{\partial \omega_x}{\partial \theta} + I_y \omega_y \frac{\partial \omega_y}{\partial \theta} + I_z \omega_z \frac{\partial \omega_z}{\partial \theta}\right) = F_\theta \tag{41}$$

$$I_x(\dot{\omega}_x \alpha_{13} + \omega_x \dot{\alpha}_{13}) + I_y(\dot{\omega}_y \alpha_{23} + \omega_y \dot{\alpha}_{23})$$
$$+ I_z(\dot{\omega}_z \alpha_{33} + \omega_z \dot{\alpha}_{33}) = F_\psi \tag{42}$$

$$I_z \dot{\omega}_z - \left(I_x \omega_x \frac{\partial \omega_x}{\partial \phi} + I_y \omega_y \frac{\partial \omega_y}{\partial \phi}\right) = F_\phi \tag{43}$$

As discussed in Ref. 2, the concept of virtual work can be employed to derive the non-conservative forces F_{z_0}, F_R, F_γ, F_θ, F_ψ, and F_ϕ which constitute the generalized forces in the Lagrange equations (38) through (43). In this technique, each of the generalized coordinates (Z_o, R, γ, θ, ψ or ϕ) are increased, one at a time by a small positive increment while holding all other coordinates fixed, and the expressions for the virtual work terms are determined. Since system forces and moments due to aerodynamic effects are applied at the cg and expressed in orthogonal X, Y coordinates, no coupling of the virtual work terms occurs and they can be

readily derived by inspection using Figs. 1 and 2. The partial derivative of the sum of the virtual work terms is then taken with respect to one of the generalized coordinates and the corresponding generalized force is thus obtained. For example:

$$\frac{\partial}{\partial z_0}\left\{F_{x_1}\left(\frac{\partial x_1}{\partial z_0}\right) + F_{y_1}\left(\frac{\partial y_1}{\partial z_0}\right) + F_{z_1}\left(\frac{\partial z_1}{\partial z_0}\right) + M_{x_1}\left(\frac{\partial \varepsilon_{x_1}}{\partial z_0}\right) + M_{y_1}\left(\frac{\partial \varepsilon_{y_1}}{\partial z_0}\right) + M_{z_1}\left(\frac{\partial \varepsilon_{z_1}}{\partial z_0}\right)\right\} dz_0 = F_{z_0} \tag{44}$$

Where ε_{x_1}, ε_{y_1}, and ε_{z_1} are positive incremental angular displacements about the respective axes X_1, Y_1, and Z_1.

Since, by inspection of ig. 1:

$$\frac{\partial z_1}{\partial z_0} = -1 \tag{45}$$

And, as a result of orthogonality:

$$\frac{\partial x_1}{\partial z_0} = \frac{\partial y_1}{\partial z_0} = \frac{\partial \varepsilon_{x_1}}{\partial z_0} = \frac{\partial \varepsilon_{y_1}}{\partial z_0} = \frac{\partial \varepsilon_{z_1}}{\partial z_0} = 0 \tag{46}$$

Thus we obtain:

$$F_{z_0} = -F_{z_1} \tag{47}$$

And, by identical procedures:

$$F_R = -F_{x_1} \tag{48}$$

$$F_\gamma = -F_{y_1} R + M_{z_1} \tag{49}$$

$$F_\theta = M_{x_1} \cos\psi + M_{y_1} \sin\psi = M_x \cos\phi + M_y \sin\phi \tag{50}$$

$$F_\phi = M_{x_1} \sin\theta \sin\psi - M_{y_1} \sin\theta \cos\psi$$
$$+ M_{z_1} \cos\theta = M_z \qquad (51)$$

$$F_\psi = M_{z_1} \qquad (52)$$

Now, by employing the direction cosine relationships listed in Table I these forces and moments can be obtained in terms of forces and moments along and about the airplane's principal axes. Thus equations (47) through (52) can be rewritten as:

$$F_{z_0} = -(\alpha_{13} F_x + \alpha_{23} F_y + \alpha_{33} F_z) \qquad (53)$$

$$F_R = -(\alpha_{11} F_x + \alpha_{21} F_y + \alpha_{31} F_z) \qquad (54)$$

$$F_\gamma = -R(\alpha_{12} F_x + \alpha_{22} F_y + \alpha_{32} F_z) + (\alpha_{13} M_x + \alpha_{23} M_y + \alpha_{33} M_z) \qquad (55)$$

$$F_\theta = M_x \cos\phi - M_y \sin\phi \qquad (56)$$

$$F_\phi = M_z \qquad (57)$$

$$F_{\psi} = (\alpha_{13} M_x + \alpha_{23} M_y + \alpha_{33} M_z) \tag{58}$$

Finally the complete dynamic equations governing motion in the chosen coordinate system can be written in terms of the six generalized coordinates z_o, R, γ, θ, ψ, and ϕ by equating the left hand terms of equations (38) through (43) with the respective right hand terms of equations (53) through (58). These equations are written in their entirety in Appendix A.

IV. INCORPORATION OF CONVENTIONAL AERODYNAMIC DATA

A. DERIVATION OF REFERENCE PARAMETERS

In order to utilize conventional aerodynamic force and moment coefficient data in the equations of motion just derived it is necessary to obtain such typical reference parameters as airplane angle of attack (α), sideslip angle (β), center of gravity velocity (V_{cg}), roll rate (p), pitch rate (q), and yaw rate (r) in terms of equation variables and their time derivatives. This can be done with the aid of Figs. 1 and 3 as follows:

$$V_{cg} = \dot{x}_1 \bar{e}_1 + \dot{y}_1 \bar{e}_2 + \dot{z}_1 \bar{e}_3 \tag{59}$$

where $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$ are unit vectors along the X_1, Y_1, Z_1 axes respectively.

But from Fig. 1:

$$\dot{x}_1 = -\dot{R}\,,\quad \dot{y}_1 = -\dot{\gamma}R,\quad \dot{z}_1 = -\dot{z}_0$$

Thus:

$$V_{cg} = -\dot{R}\,\bar{e}_1 - \dot{\gamma}R\,\bar{e}_2 - \dot{z}_0\,\bar{e}_3 \tag{60}$$

Therefore the relative wind at the airplane cg is:

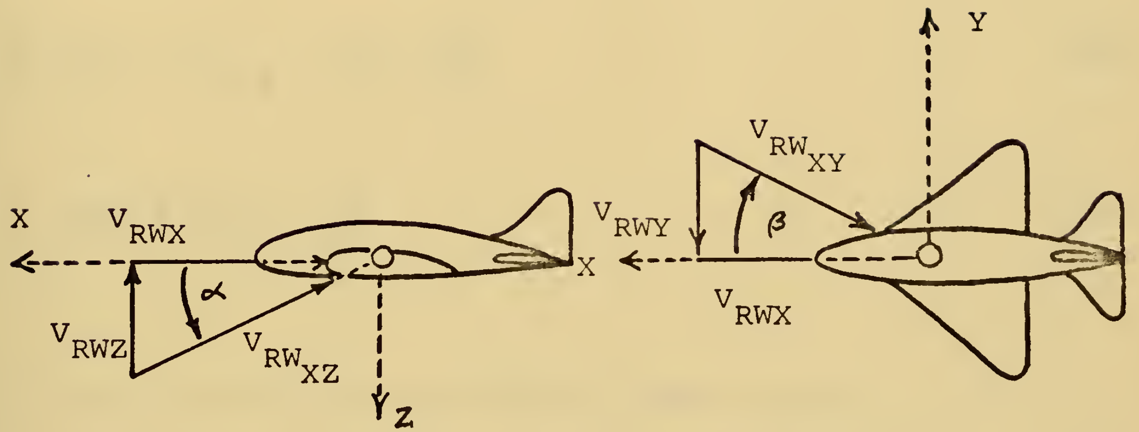
$$V_{RW} = \dot{R}\,\bar{e}_1 + \dot{\gamma}R\,\bar{e}_2 + \dot{z}_0\,\bar{e}_3 \tag{61}$$

By use of the direction cosines given in Table I, the relative wind can be resolved into components along the airplane's X, Y and Z axes as follows:

$$V_{RW_X} = \alpha_{11}\dot{R} + \alpha_{12}\dot{\gamma}R + \alpha_{13}\dot{z}_0 \tag{62}$$

$$V_{RW_Y} = \alpha_{21}\dot{R} + \alpha_{22}\dot{\gamma}R + \alpha_{23}\dot{z}_0 \tag{63}$$

$$V_{RW_Z} = \alpha_{31}\dot{R} + \alpha_{32}\dot{\gamma}R + \alpha_{33}\dot{z}_0 \tag{64}$$



$$\tan\alpha = \frac{V_{RWZ}}{V_{RWX}} \qquad \tan\beta = \frac{V_{RWY}}{V_{RWX}}$$

Fig. 3 Definition of Positive α and β Angles

Therefore α and β can be written as:

$$\alpha = \tan^{-1}\left(\frac{\alpha_{31}\dot{R} + \alpha_{32}\dot{\gamma}R + \alpha_{33}\dot{z}_0}{\alpha_{11}\dot{R} + \alpha_{12}\dot{\gamma}R + \alpha_{13}\dot{z}_0}\right) \tag{65}$$

$$\beta = \tan^{-1}\left(\frac{\alpha_{21}\dot{R} + \alpha_{22}\dot{\gamma}R + \alpha_{23}\dot{z}_0}{\alpha_{11}\dot{R} + \alpha_{12}\dot{\gamma}R + \alpha_{13}\dot{z}_0}\right) \tag{66}$$

The magnitude of the relative wind can be written from equation (61) as:

$$V_{RW} = \left\{\dot{R}^2 + (\dot{\gamma}R)^2 + \dot{z}_0^2\right\}^{1/2} \tag{67}$$

Finally, the rotational rates p , q , and r about the airplane's x, y , and z axes are identical to the angular rates ω_x , ω_y and ω_z given by equations (3), (4), and (5), and can be written directly as:

$$p = (\dot{\psi} + \dot{\gamma})\alpha_{13} + \dot{\theta}\cos\phi \tag{68}$$

$$q = (\dot{\psi} + \dot{\gamma})\alpha_{23} - \dot{\theta}\sin\phi \tag{69}$$

$$r = (\dot{\psi} + \dot{\gamma})\alpha_{33} + \dot{\phi} \tag{70}$$

B. MODELING THE FORCES FOR THE STEADY SPIN

Tabulated aerodynamic data, although virtually always appearing in the form of dimensionless coefficients, varies widely in terms of the reference axis system, point of application of forces and moments, and the characteristic length used in non-dimensionalization, depending on the intended application of the data. By the use of the principal axis system in the foregoing derivation of the equations of motion, considerable simplication was achieved. Therefore it appears that conversion of aerodynamic data to the principal axis system and cg as references would generally be preferable to the alternative of adapting the equations to the particular form of the data. This should present no serious problems, however, since the equations must be solved numerically and conversion of the data by computer

is by comparison a relatively minor additional task. Since some data manipulation will likely be required in any event, the mean aerodynamic chord ($\bar{C}$) has been used in the following development as a common characteristic length in an attempt to be consistent and also allow the use of a single "characteristic" length parameter as a computer program input. However, the equations which follow could be adjusted appropriately to use other characteristic lengths.

The general form of the aerodynamic forces and monents contained in the right hand terms of equations (53) through (58) can be expressed as follows:

$$F_i = \tfrac{1}{2} C_{F_i} \rho V^2 S \qquad i = x, y, \text{ or } z$$

$$M_i = \tfrac{1}{2} C_{M_i} \rho V^2 S \bar{C}$$

Where $\bar{c}$ has been used as a characteristic length in non-dimensionalizing all moment coefficients. The force and moment coefficients (C_{F_i} and C_{M_i}) are, in turn, of the following general form:

$$C_{F_i} = C_{F_{i_1}} + C_{F_{i_2}} \cdot f(p) + C_{F_{i_3}} \cdot f(q) + C_{F_{i_4}} \cdot f(r) + C_{F_{i_5}} \cdot f(\delta_a) + C_{F_{i_6}} \cdot f(\delta_r) + C_{F_{i_7}} \cdot f(\delta_h)$$

$$C_{M_i} = C_{M_{i_1}} + C_{M_{i_2}} \cdot f(p) + C_{M_{i_3}} \cdot f(q) + C_{M_{i_4}} \cdot f(r) + C_{M_{i_5}} \cdot f(\delta_a) + C_{M_{i_6}} \cdot f(\delta_r) + C_{M_{i_7}} f(\delta_h)$$

Where:

1. All $C_{F_{ij}}$ and $C_{M_{ij}}$ (j = 1,2,···7 and i = x,y or z) are aerodynamic force and moment derivatives, as

defined in Table II below, which must be experimentally determined and tabulated for a wide range of α and β values.

Table II: Definitions of Aerodynamic Derivatives[5]

	SUBSCRIPT j						
i=x,y, or z	1	2	3	4	5	6	7
C_{Fij}	$C_{Fi}(\alpha,\beta)$	$\frac{\partial C_{Fi}}{\partial(\frac{p\bar{c}}{2V})}$	$\frac{\partial C_{Fi}}{\partial(\frac{q\bar{c}}{2V})}$	$\frac{\partial C_{Fi}}{\partial(\frac{r\bar{c}}{2V})}$	$\frac{\partial C_{Fi}}{\partial \delta_a}$	$\frac{\partial C_{Fi}}{\partial \delta_r}$	$\frac{\partial C_{Fi}}{\partial \delta_h}$
C_{Mij}	$C_{Mi}(\alpha,\beta)$	$\frac{\partial C_{Mi}}{\partial(\frac{p\bar{c}}{2V})}$	$\frac{\partial C_{Mi}}{\partial(\frac{q\bar{c}}{2V})}$	$\frac{\partial C_{Mi}}{\partial(\frac{r\bar{c}}{2V})}$	$\frac{\partial C_{Mi}}{\partial \delta_a}$	$\frac{\partial C_{Mi}}{\partial \delta_r}$	$\frac{\partial C_{Mi}}{\partial \delta_h}$

2. $f(p)$, $f(q)$, and $f(r)$ are functions of roll, pitch and yaw rate respectively.

3. δ_a, δ_r and δ_h are defined as:

   δ_a = Aileron deflection

   δ_r = Rudder deflection

   δ_h = Elevation deflection

A good example of the form of tabular data that could be used in this manner is contained in Ref. 3.

At this point some comments are appropriate to insure that the above method of handling the aerodynamic forces and monents are put in proper perspective. First of all it should be noted that the validity of any calculations of spin parameters obtained by this approach to the spin

[5]Aerodynamic derivatives with respect to $\dot{\alpha}$ and $\dot{\beta}$ have not been included since $\dot{\alpha}$ and $\dot{\beta}$ are zero for the steady spin.

problem depends entirely on the modeling of the aerodynamic forces and monents since the equations of motion as derived in Section III are exact. Secondly it should be recognized that the aerodynamics of the spin problem involve low speed three dimensional stalled flow, and as such forces and moments are nonlinear. Thus the superposition of the separate effects of triaxial rotation, control deflection, and other static wind tunnel data cannot be justified analytically. The only wind tunnel test technique that presently appears to offer any possibility of yielding satisfactory aerodynamic data for spin is the rotary balance force and moment measuring apparatus which is reported on in Ref. 4.

Next, allowing that for want of better data the conventional approach outlined above will be used to exercise computer program and look for a zeroth order approximation of spin characteristics, there is another problem that stems from this application of conventional data. The problem is that existing data is generally of insufficient range in α and β to accommodate the high angles of attack and sideslip angles characteristic of autogyration. Also many of the basic set of 42 aerodynamic coefficients suggested by the above development describe coupling effects which in conventional airplanes at controlled flight altitudes are, in general, not significant and therefore have not been measured. Thus, for example, roll moment due to elevator deflection is normally ignored, but may be of some significance in a spinning F4 airplane where the strong negative

dihedral of the stabalator could cause the upwind panel to act as a rudder with prospin deflection at high α and β angles. Certainly in such cases none of the aerodynamic derivatives including the 42 above can be ignored without testing.

Now admitting to the fact that force modeling with conventional stability derivatives will be done with at best an incomplete set of data, it then becomes necessary to exercise caution to insure that those coefficients that are available and used will allow a force and moment balance to be somehow achieved. Thus, if at least one damping term is not included for every driving term in each of the six degrees of freedom, obviously no stabilized solution is attainable. Finally, in fairness to this effort it must be pointed out that the modeling of aerodynamic forces and moments on the spinning airplane, although recognized as the key to the problem under study, was non-the-less peripheral to the expressed purpose of this preliminary work. Once a workable computer program has been written and checked by use of conventional aerodynamic data, subsequent efforts should be directed towards improving the modeling of aerodynamic forces and moments.

V. SOLUTION OF THE EQUATIONS OF MOTION

A. GENERAL CONCEPT

From an inspection of the equations of motion contained in Appendix A (in abbreviated format) it is evident that, although they are ordinary second order differential equations, they are also so extensively coupled and non-linear that direct analytical integration is out of the question. Even if the left hand sides of the equations were linear, the forces and moments which constitute the right hand sides are functions of a three dimensional flow involving the geometry of the airplane. Consequently any solution scheme will of necessity involve some form of iterative numerical procedure to circumvent direct integration. The problem thus becomes one of finding an iterative scheme which provides some degree of uniform convergence to steady or quasi-steady values of the six spin parameters ($\dot{z}_o$, R , $\dot{\gamma}$, θ , ψ , and ϕ) without losing numerical significance or using excessive computer time.

If the problem is visualized as a trial and error positioning and orientation problem where the airplane is incrementally reoriented angularly in response to an aerodynamic moment inbalance and simultaneously adjusted in $\dot{z}_o$, $\dot{\gamma}$ and R in response to an aerodynamic force inbalance, it then appears to be reducible to a simple series of stepwise adjustments or iterations. Also, since the problem would thus be restarted on each successive iteration with updated

force and moment data corresponding to the adjusted spin parameters, such a procedure would avoid the possibility of loosing numerical significance. The problem of determining consistent incremental changes in each spin parameter to achieve some degree of uniform convergence of such an iterative scheme can be easily handled by using the principle of conservation of momentum. By rearranging the equations of motion in such a way as to facilitate the computation of accelerations in each of the six spin coordinates it is then possible to apply a common increment of time and thereby compute a dynamically consistent set of incremental changes in the parameters and obtain new parameter values for the subsequent iteration. This, in a sense, amounts to following the motion of the airplane as is done in the conventional moving coordinate system approach to the problem, but the hybrid inertial coordinate system has several distinct advantages in this application, as discussed below.

The rationale upon which this suggested solution technique is based is that it duplicates in a computational sense, what actually happens to an airplane under such flight conditions. However one key advantage that the computational scheme has over the actual airplane is that, on each restart or iteration we can discard any angular momentum inconsistent, with a steady spin situation. Thus, we can avoid the problem of carrying along tumbling momentum which could considerably lengthen the computational time or cause the solution to pass right through a steady spin mode or recovery without

stabilizing. Of course to realize this advantage it is necessary to be able to differentiate between that part of the motion which constitutes tumbling and that part which is consistent with steady spinning. Herein lies one of the major advantages of writing the equations of motion in an inertial cylindrical reference frame rather than in a body-fixed moving coordinate system, for in the hybrid inertial system tumbling is readily recognizable as terms involving $\dot{\theta}, \dot{\psi}$ or $\dot{\phi}$.

B. SIMPLIFICATION OF THE EQUATIONS OF MOTION

With the above approach in mind the equations of motion can now be simplified sufficiently to meet the requirements of the solution scheme without the necessity of making any restrictive simplifying assumptions. Also since the intended iterative technique deals with the problem directly in dynamical units, we do not have to non-dimensionalize as would be necessary if one resorted to a somewhat more abstract mathematical method of varying the spin parameters to obtain convergence. As a result one can avoid obscuring the physical nature of the problem and is in a better position to utilize insight in monitoring, trouble shooting, and interpreting iteration trends.

The simplification is started by recognizing that only steady or quasi-steady spin modes are sought. Thus, by setting angular orientation rates ($\dot{\theta}$, $\dot{\psi}$, and $\dot{\phi}$) equal to zero a considerable number of terms can at once be eliminated. The equations contained in Appendix A thus reduce to the following form:

Z_0 Equation:

$$M\ddot{Z}_0 + Mg = -\left(C_{F_X}\alpha_{13} + C_{F_Y}\alpha_{23} + C_{F_Z}\alpha_{33}\right)\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\} \tag{73}$$

R Equation:

$$M\ddot{R} - M\dot{\gamma}^2 R = -\left(C_{F_X}\alpha_{11} + C_{F_Y}\alpha_{21} + C_{F_Z}\alpha_{31}\right)\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\} \tag{74}$$

γ Equation:

$$\begin{aligned} &MR(2\dot{R}\dot{\gamma} + R\ddot{\gamma}) + I_X\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{13} + \ddot{\theta}\cos\phi\right\}\alpha_{13} + I_Y\left\{(\ddot{\psi}\right. \\ &\left.+\ddot{\gamma})\alpha_{23} - \ddot{\theta}\sin\phi\right\}\alpha_{23} + I_Z\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{33} + \ddot{\phi}\right\}\alpha_{33} \\ &= \left\{\bar{c}\left(C_{M_X}\alpha_{13} + C_{M_Y}\alpha_{23} + C_{M_Z}\alpha_{33}\right) - R\left(C_{F_X}\alpha_{12}\right.\right. \\ &\left.\left.+ C_{F_Y}\alpha_{22} + C_{F_Z}\alpha_{32}\right)\right\}\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\} \end{aligned} \tag{75}$$

θ Equation:

$$I_x\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{13}+\ddot{\theta}\cos\phi\right\}\cos\phi-I_y\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{23}-\ddot{\theta}\sin\phi\right\}\sin\phi-I_x\dot{\gamma}^2\cos\theta\sin\phi\,\alpha_{13}$$
$$-I_y\dot{\gamma}^2\cos\theta\cos\phi\,\alpha_{23}+I_z\dot{\gamma}^2\sin\theta\,\alpha_{33}$$
$$=(C_{M_X}\cos\phi-C_{M_Y}\sin\phi)\frac{\rho S\bar{c}}{2}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_o)^2\right\} \tag{76}$$

ψ Equation:

$$I_x\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{13}+\ddot{\theta}\cos\phi\right\}\alpha_{13}+I_y\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{23}-\ddot{\theta}\sin\phi\right\}\alpha_{23}$$
$$+I_z\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{33}+\ddot{\phi}\right\}\alpha_{33}$$
$$=(C_{MX}\alpha_{13}+C_{MY}\alpha_{23}+C_{M_Z}\alpha_{33})\frac{\rho S\bar{c}}{2}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_o)^2\right\} \tag{77}$$

ϕ Equation:

$$I_z\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{33}+\ddot{\phi}\right\}-I_x\dot{\gamma}^2\sin\theta\cos\phi\,\alpha_{13}$$
$$+I_y\dot{\gamma}^2\sin\theta\sin\phi\,\alpha_{23}$$
$$=(C_{M_Z})\frac{\rho S\bar{c}}{2}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_o)^2\right\} \tag{78}$$

Although $\dot{R}$ is a rate parameter which is not consistent with the steady spin assumption, it effects both the values of α and β and also the magnitude of the relative wind, and therefore must be retained in the equations.

Next, to put the equations into a useable form for the suggested iteration scheme we simply solve each equation

for the corresponding acceleration parameter in terms of the other variables. The equations then become:

Z Equation:

$$\ddot{Z}_0 = -g - (C_{F_x}\alpha_{13} + C_{F_y}\alpha_{23} + C_{F_z}\alpha_{33})\frac{\rho S}{2M}\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\} \tag{79}$$

R Equation:

$$\ddot{R} = \dot{\gamma}^2 R - (C_{F_x}\alpha_{11} + C_{F_y}\alpha_{21} + C_{F_z}\alpha_{31})\frac{\rho S}{2M}\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\} \tag{80}$$

γ Equation:

$$\ddot{\gamma} = \{MR^2 + I_x\alpha_{13}^2 + I_y\alpha_{23}^2 + I_z\alpha_{33}^2\}^{-1}\Big[-(I_x\alpha_{13}^2 + I_y\alpha_{23}^2 + I_z\alpha_{33}^2)\ddot{\psi} - (I_x - I_y)\ddot{\theta}\,SIN\,\theta\,SIN\,\phi\,COS\,\phi - I_z\ddot{\phi}\alpha_{33} - 2MR\dot{R}\dot{\gamma} + \{\bar{c}(C_{M_x}\alpha_{13} + C_{M_y}\alpha_{23} + C_{M_z}\alpha_{33}) - R(C_{F_x}\alpha_{12} + C_{F_y}\alpha_{22} + C_{F_z}\alpha_{32})\}\frac{\rho S}{2}\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\}\Big] \tag{81}$$

θ Equation:

$$\ddot{\theta} = \{I_x COS^2\phi + I_y SIN^2\phi\}^{-1}\Big[-(I_x - I_y)(\ddot{\psi} + \ddot{\gamma})\,SIN\,\theta\,SIN\,\phi\,COS\,\phi + (I_x SIN^2\phi + I_y COS^2\phi - I_z)\dot{\gamma}^2\,SIN\,\theta\,COS\,\theta + \{C_{M_x} COS\,\phi - C_{M_y} SIN\,\phi\}\frac{\rho S\bar{c}}{2}\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\}\Big] \tag{82}$$

ψ Equation:

$$\ddot{\psi}=\left\{I_x\alpha_{13}^2+I_y\alpha_{23}^2+I_z\alpha_{33}^2\right\}^{-1}\Big[-(I_x\alpha_{13}^2+I_y\alpha_{23}^2+I_z\alpha_{33}^2)\ddot{\gamma}$$
$$-(I_x-I_y)\ddot{\theta}\,SIN\,\theta\,SIN\,\phi\,COS\,\phi-I_z\ddot{\phi}\,COS\,\theta+\left\{C_{M_x}\alpha_{13}\right. \tag{83}$$
$$\left.+C_{M_y}\alpha_{23}+C_{M_z}\alpha_{33}\right\}\frac{\rho S\bar{c}}{2}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_o)^2\right\}\Big]$$

ϕ Equation:

$$\ddot{\phi}=\left(\frac{I_x-I_y}{I_z}\right)\dot{\gamma}^2SIN^2\theta\,SIN\,\phi\,COS\,\phi-(\ddot{\psi}+\ddot{\gamma})\,COS\,\theta \tag{84}$$
$$+C_{M_z}\frac{\rho S\bar{c}}{2I_z}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_o)^2\right\}$$

Before proceeding to develop the details of the proposed iterative scheme of solving these equations, some comments on the steady spin assumption should be made. First of all, truly oscillatory spin modes, as opposed to lightly damped steady spin modes that may be very slow in convergence should not, in general, result from the use of mean force and moment model test date, the usual form of such data. Thus if the oscillatory nature of a spin were attributable to periodic vortex shedding from the fuselage or stalled aerodynamic surfaces (von Karman Vortex Street phenomenon), the use of mean aerodynamic forces and moments should effectively filter out such oscillation from the computer solution. If however the oscillations were strictly a result of the dynamics of the problem, such as a limit cycle type of oscillation, then the oscillatory nature of the spin should appear in the computations as a failure to

converge to steady spin parameter values. In this case, the complete equations would have to be used along with a very small time increment and the stability derivatives due to $\dot{\alpha}$ and $\dot{\beta}$ to obtain an accurate description of the motion in the form of a numerical time history. Although this would involve solving the full equations for the corresponding acceleration terms as was done in equations (79) through (84) and would also lengthen the computation time it would not be a difficult extension and might well provide some valuable insight into the stability of such oscillatory modes. This form of the spin equations would be especially valuable in studying the effect of applying oscillatory recovery control inputs in resonance with the spin oscillations to obtain an "autopilot spin recovery" from an otherwise irrecoverable spin mode. Another use of the full equations would be to study the stability of spin solutions obtained by other numerical methods. However terms involving $\dot{\theta}$, $\dot{\psi}$, and $\dot{\phi}$ would re-introduce the effects of angular momentum into the computations and would probably cause excessive tumbling if the full equations were used to initially solve for spin modes. Therefore either the following computational scheme or some other numerical technique for solving non-linear differential equations such as the iterative procedure described on page 270 of Ref. 7 should be employed to initially solve for spin modes.

C. SUGGESTED COMPUTER SOLUTION SCHEME

Although the equations are rather complex the suggested computer iteration scheme is fairly simple. Basically some

starting position or initial "guess" must be supplied in terms of altitude rate, spin radius[6], spin rate, orientation (θ, ψ, and ϕ), along with control positions δ_a, δ_r and δ_h. Angle of attack and angle of sideslip are then computed from equations (65) and (66) and, in turn α and β are used to obtain values for the various aerodynamic derivatives discussed in Section IV B by interpolating from appropriate tables. The six aerodynamic force and moment coefficients, C_{F_i} and C_{M_i} are then computed as outlined in equations (71) and (72) and, with these values in hand, $\ddot{z}_o$ and $\ddot{R}$ can be computed directly from equations (79) and (80). The other accelerations $\ddot{\gamma}$, $\ddot{\theta}$, $\ddot{\psi}$, and $\ddot{\phi}$, can be computed by a looping routine encompassing equations (81) through (84). This looping will be necessary due to the coupling of the respective acceleration terms in each of these equations. Once consistent values of $\ddot{\gamma}$, $\ddot{\theta}$, $\ddot{\psi}$, and $\ddot{\phi}$ have been obtained from the looping iteration, incremental changes in each of the seven computation parameters can be obtained from the following equations:

$$\dot{z}_{o(n+1)} = \dot{z}_{o(n)} + \ddot{z}_o \Delta t \tag{85}$$

$$\dot{R}_{(n+1)} = \dot{R}_{(n)} + \ddot{R} \Delta t \tag{86}$$

$$R_{(n+1)} = R_{(n)} + \dot{R}_{(n+1)} \Delta t + \ddot{R} \frac{\Delta t^2}{2} \tag{87}$$

---

[6] Radius rate, $\dot{R}$, is initially set equal to zero.

$$\dot{\gamma}_{(n+1)} = \dot{\gamma}_{(n)} + \ddot{\gamma}\Delta t \tag{88}$$

$$\theta_{(n+1)} = \theta_{(n)} + \ddot{\theta}\,\frac{\Delta t^2}{2} \tag{89}$$

$$\psi_{(n+1)} = \psi_{(n)} + \ddot{\psi}\,\frac{\Delta t^2}{2} \tag{90}$$

$$\phi_{(n+1)} = \phi_{(n)} + \ddot{\phi}\,\frac{\Delta t^2}{2} \tag{91}$$

If the full equations were to be used, obviously it would then be necessary to also compute $\dot{\theta}$, $\dot{\psi}$, $\dot{\phi}$, θ, ψ, and ϕ as follows:

$$\dot{\theta}_{(n+1)} = \dot{\theta}_{(n)} + \ddot{\theta}\Delta t \tag{92}$$

$$\theta_{(n+1)} = \theta_{(n)} + \dot{\theta}_{(n+1)}\Delta t + \ddot{\theta}\,\frac{\Delta t^2}{2} \tag{93}$$

$$\dot{\psi}_{(n+1)} = \dot{\psi}_{(n)} + \ddot{\psi}\Delta t \tag{94}$$

$$\psi_{(n+1)} = \psi_{(n)} + \dot{\psi}_{(n+1)}\Delta t + \ddot{\psi}\,\frac{\Delta t^2}{2} \tag{95}$$

$$\dot{\phi}_{(n+1)} = \dot{\phi}_{(n)} + \ddot{\phi}\Delta t \tag{96}$$

$$\phi_{(n+1)} = \phi_{(n)} + \dot{\phi}_{(n+1)}\Delta t + \ddot{\phi}\frac{\Delta t}{2} \tag{97}$$

The computational sequence outlined above is then repeated using the revised values of the parameters involved and continued until some pre-designated convergence criteria is met, such as all accelerations being less than some small epsilon value or the rates of change of the various parameters are each in turn less than some small characteristic values as individually specified. For example, one foot per second in $\dot{R}$, 0.5 feet per second in $\dot{z}_o$, one degree per second in $\dot{\gamma}$, etc.

The choice of an incremental time probably will have a very significant effect on the overall success of this iteration scheme. The use of a very small value of Δt will of course provide the smoothest convergence but at a significant expense in computer time. A large Δt on the other hand can easily preclude convergence by drastically altering all parameters from iteration to iteration. To illustrate this point and also pin down what is meant by large and small Δt it is necessary to step back a bit and consider conceptually what is really being done by using an iterative solution scheme. As pointed out earlier, the equations are highly non-linear and extensively coupled, and consequently one has very little insight as to what linearizing approximations would be valid or, for that matter, over what range of values can the effects of an

individual parameter change be considered linear. This problem is however overcome in the suggested iteration scheme by allowing the dynamics of the problem to pick the magnitude of the incremental changes in each parameter. Thus if the Δt used is of such a magnitude as to constitute a small pertubation relative to the characteristic time of the problem, the time per revolution, we essentially establish appropriately sized small pertubations in each of the spin parameters as well. Thus, for example, a spin rate of 40° per second would yield a 9 second turn and possibly a good starting guess for an appropriately sized Δt would be .09 seconds. The validity of this concept will not be rigorously or otherwise established here, other than to say that in general small pertubation techniques have enjoyed some success in other engineering applications and this appears to be a quite natural extension.

The problem of picking appropriate starting guesses for $\dot{z}_0$, R, $\dot{\gamma}$, θ, ψ, and ϕ unfortunately is not as easily handled as deciding on an initial Δt. Although some rather broad guidelines can be stated such as steep nose down pitch, large radius and low spin rate for normal erect (upright) spins and shallow nose down pitch, small radius and high spin rate for flat spins, the actual searching for individual modes must proceed on a trial and error basis until an analysis of solutions suggests some better procedures. Starting guesses that don't lead to spins will of course show up as recoveries to linear stalled flight attitudes or complete fly-away recoveries.

Certainly some insight can be used to pick $\dot{z}_o$ as a negative (down) and of an order of magnitude consistent with stalled flight velocities (T.A.S.) at the altitude where the solution is sought. Likewise, a wings level attitude with the nose down and pointed at the spin axis, e.g., $\theta = +30^{\circ}$, $\psi = -90^{\circ}$ and $\phi = +90^{\circ}$, would be a reasonable starting orientation for an erect right spin. Finally, $\dot{\gamma}$ should obviously be chosen small so that it will not toss the airplane out of the ball park under the action of centrifugal force before the aerodynamics can come into play. One point should be made however concerning searches for flat spin modes with small radius values. If conventional force modeling is used, as discussed in Section IV B, then in addition to the general lack of validity of adding non-linear aerodynamic forces and moments, there is the compounding of errors due to the fact that free stream α, V and β vary from point to point over the fuselage and wings as a function of orientation when radius is reduced to the same order of magnitude as fuselage length or wing span. Thus, for example, if the wings are level and the nose is pointed at the spin axes in a fairly flat attitude the geometry of the problem will yield a significant difference in both sideslip velocity, and consequently β, from nose to tail for a small spin radius. Of course aerodynamic data for this type of flow condition cannot be obtained in a static wind tunnel setup.

D. AUXILIARY ANGULAR RELATIONSHIPS

The particular set of "Euler Angles" used in the derivation of equations (79) through (84) is just one of several variations of such positioning angles which are generally classified as Euler Angles. The principal advantage gained by the use of such sets of positioning angles is that they describe a single fixed spatial orientation of a rigid body for given values of θ, ψ, and ϕ, and yield a convenient set of equations for the direction cosines with the order of angular rotation implicit in the equations. However unlike the more conventional pitch, roll and yaw orientation angles which can be readily visualized once the order is specified, orientation in terms of Euler Angles is difficult to visualize without some sketching, which can be readily appreciated by refering to Fig. 2. Thus unless the orientation is close to quadrantal a computer output of orientation in Euler Angles may be difficult to interpret.

To alleviate this situation, relationships between an ordered set of orientation angles (1) pitch, (2) roll, and (3) yaw, and the Euler angles shown in Fig. 2 were derived. The derivation was facilitated by the sketch of the spherical triangle relationships between the two angular sets shown in Fig. 4 and involves the use of Napier's analogies from Spherical Trigonometry, Ref. 5.

The use of Napier's analogies avoids the quadrantal ambiguity associated with the solution of right spherical

triangles by other methods. Thus the equations for converting the ordered angular rotations θ_P, ϕ_R and ψ_Y to the set of Euler angles used in this paper are:

$$\phi = \psi_Y - \mathrm{Tan}^{-1}\left\{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)\mathrm{Tan}\left(\frac{\phi_R}{2}\right)\right\} - \mathrm{Tan}^{-1}\left\{\frac{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)}{\mathrm{Tan}\left(\frac{\phi_R}{2}\right)}\right\} \tag{98}$$

$$\psi = \mathrm{Tan}^{-1}\left\{\frac{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)}{\mathrm{Tan}\left(\frac{\phi_R}{2}\right)}\right\} - \mathrm{Tan}^{-1}\left\{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)\mathrm{Tan}\left(\frac{\phi_R}{2}\right)\right\} \tag{99}$$

$$\theta = 2\,\mathrm{Tan}^{-1}\left\{\frac{\mathrm{Sin}\left[\mathrm{Tan}^{-1}\left\{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)\mathrm{Tan}\left(\frac{\phi_R}{2}\right)\right\}\right]}{\mathrm{Sin}\left[\mathrm{Tan}^{-1}\left\{\frac{\mathrm{Tan}\left(\frac{\theta_P}{2}\right)}{\mathrm{Tan}\left(\frac{\phi_R}{2}\right)}\right\}\right]\mathrm{Tan}\left(\frac{\phi_R}{2}\right)}\right\} \tag{100}$$

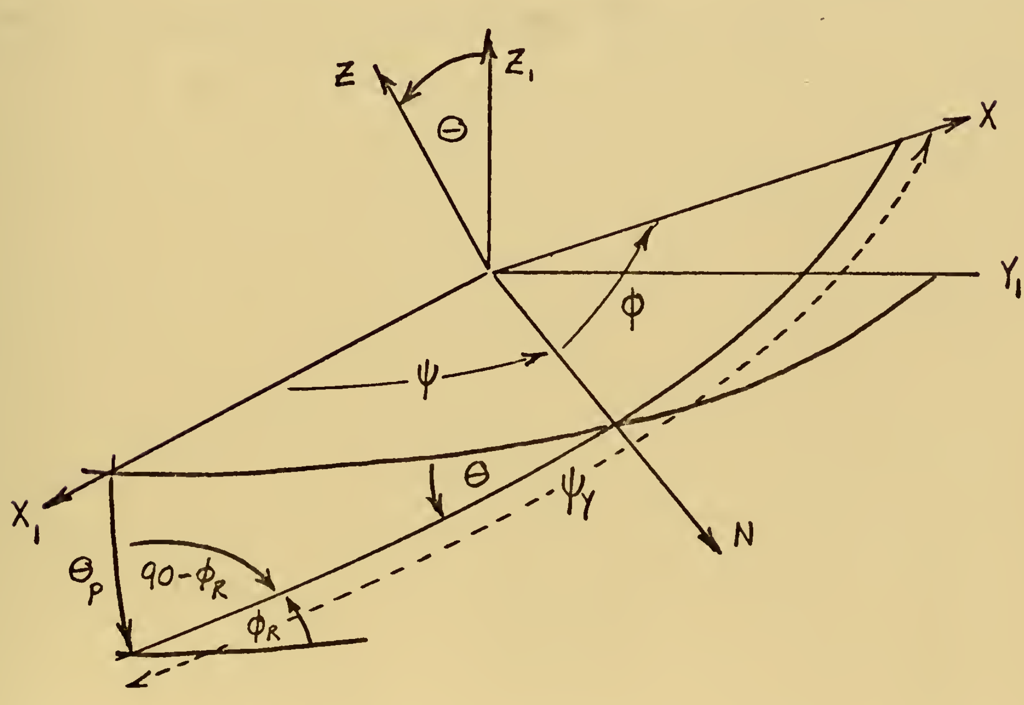


Fig. 4 Schematic of Auxiliary Angular Relationships

The equations for obtaining θ_P, ϕ_R and ψ_Y from Euler angles are:

$$\psi_Y = \phi + \tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\sin\left(\frac{90-\theta}{2}\right)}{\sin\left(\frac{90+\theta}{2}\right)}\right\} + \tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\cos\left(\frac{90-\theta}{2}\right)}{\cos\left(\frac{90+\theta}{2}\right)}\right\} \quad (101)$$

$$\theta_P = \tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\cos\left(\frac{90-\theta}{2}\right)}{\cos\left(\frac{90+\theta}{2}\right)}\right\} - \tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\sin\left(\frac{90-\theta}{2}\right)}{\cos\left(\frac{90+\theta}{2}\right)}\right\} \quad (102)$$

$$\phi_R = 90 - 2\tan^{-1}\left\{\frac{\sin\left[\tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\sin\left(\frac{90-\theta}{2}\right)}{\sin\left(\frac{90+\theta}{2}\right)}\right\}\right]}{\sin\left[\tan^{-1}\left\{\frac{\tan\left(\frac{\psi}{2}\right)\cos\left(\frac{90-\theta}{2}\right)}{\cos\left(\frac{90+\theta}{2}\right)}\right\}\right]\tan\left(\frac{90-\theta}{2}\right)}\right\} \quad (103)$$

Equations (101) through (103) can be converted for computations in radians by substituting $\pi/2$ for 90°.

VI. THE INVERSE PROBLEM

As discussed in Section III B the conventional method of modeling the aerodynamic forces and moments on the spinning airplane with static wind tunnel data lacks mathematical validity. Therefore it will be necessary to develop a better method of modeling the aerodynamics of the problem if the computational approach to predicting spin modes is to be successful. The work done with the rotary balance, Ref. 4 has been one effort in this area. Such work, however, is difficult to pursue without some method of obtaining the actual forces and moments as a basis for comparison. Apparently in the past the validity of such aerodynamic data has been checked by the indirect method of using it to solve the partially linearized equation and then comparing computed spin modes with known spin characteristics. This approach may allow some estimate of the overall success of the combined result of the equations and data but doesn't establish the validity of either. A better approach would be to use the exact equations derived in Section V in such a manner as to allow comparison of the 6 computed aerodynamic forces and moments with the corresponding forces and moments on the actual spinning airplane. Fortunately this can be done for the steady spin case and thus allow some progress to be made in modeling steady or mean values of forces and moments.

In order to use the equations of motion derived in Section V in this application it is first necessary to obtain the values of $\dot{z}_0$, R, Θ, ψ, and ϕ for the actual airplane or free spinning model. For the case of the model in the free-spinning wind tunnel discussed in Ref. 6, determination of the above spin parameters can obviously be obtained from tunnel speed, and photographic coverage, and thus will not be discussed further. For the actual airplane the means of getting these parameters is still fairly simple but in some cases less obvious, and consequently will be outlined here. First of all, $\dot{z}_0$ can be determined by the pilot altitude/time observation or photo panel data, or externally by ground station precision radar or optical tracking. Likewise $\dot{\gamma}$ can be determined either by the pilot or from ground station photographic coverage. Spin radius can be obtained by resolving triaxial cg accelerometer data along the principal axes and using the following relationships:

$$g_x^2 + g_y^2 + g_z^2 = 1.0 + (\dot{\gamma}^2 R)^2 \tag{104}$$

$$R = \frac{1}{\dot{\gamma}^2}\sqrt{g_x^2 + g_y^2 + g_z^2 - 1} \tag{105}$$

Airplane orientation in terms of Euler Angles Θ, ψ, and ϕ can be determined from values of pitch, roll, and yaw rates and the triaxial accelerations as follows:

$$P = \dot{\gamma} \sin\Theta \sin\phi \tag{106}$$

$$q = \dot{\gamma} \sin\theta \cos\phi \tag{107}$$

$$r = \dot{\gamma} \cos\theta \tag{108}$$

Thus:

$$\phi = \tan^{-1}\left\{\frac{p}{q}\right\} \tag{109}$$

$$\theta = \tan^{-1}\left\{\frac{p}{r \sin\phi}\right\} \tag{110}$$

and from Table I:

$$g_{x_1} = \sqrt{g_x^2 + g_y^2 + g_z^2 - 1} = g_x \alpha_{11} + g_y \alpha_{21} + g_z \alpha_{31}$$

or solving for ψ :

$$\psi = \cos^{-1}\left\{\sqrt{\frac{g_x^2 + g_y^2 + g_z^2 - 1}{(g_x \cos\phi - g_y \sin\phi)^2 + (g_x \sin\phi \cos\theta + g_y \cos\phi \cos\theta - g_z \sin\theta)^2}}\right\} \tag{111}$$

$$- \tan^{-1}\left\{\frac{g_x \sin\phi \cos\theta + g_y \cos\phi \cos\theta - g_z \sin\theta}{g_x \cos\phi - g_y \sin\phi}\right\}$$

Obviously there will be some loss of accuracy in the determination of spin radius and orientation if the spin is not exactly steady. However small fluctuations in the triaxial accelerations and pitch, roll and yaw rates can be averaged out and should allow computation of representative values of R, θ, ψ, and ϕ for the near steady spin case.

Next, equations (79) through (84) can be adapted to the problem at hand by setting $\dot{R}$ and all acceleration parameters equal to zero. The equations thus reduce to the following:

$\ddot{z}$ Equation:

$$-\frac{2gM}{\rho S\left\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\right\}} = C_{F_x}\alpha_{13} + C_{F_Y}\alpha_{23} + C_{F_z}\alpha_{33} \tag{112}$$

R Equation:

$$\frac{2\dot{\gamma}^2RM}{\rho S\left\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\right\}} = C_{F_x}\alpha_{11} + C_{F_Y}\alpha_{21} + C_{F_z}\alpha_{31} \tag{113}$$

γ Equation:

$$R(C_{F_x}\alpha_{12} + C_{F_y}\alpha_{22} + C_{F_z}\alpha_{32}) = \bar{C}(C_{M_x}\alpha_{13} + C_{M_y}\alpha_{23} + C_{M_z}\alpha_{33}) \tag{114}$$

θ Equation:

$$-\frac{2(I_x \sin^2\phi + I_y \cos^2\phi - I_z)\dot{\gamma}^2 \sin\theta\cos\theta}{\rho S\bar{C}\left\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\right\}} = C_{M_x}\cos\phi - C_{M_y}\sin\phi \tag{115}$$

ψ Equation:

$$C_{M_x}\alpha_{13} + C_{M_y}\alpha_{23} + C_{M_z}\alpha_{33} = 0 \tag{116}$$

ϕ Equation:

$$-\frac{2(I_x - I_y)\dot{\gamma}^2 \sin^2\theta \sin\phi\cos\phi}{\rho S\bar{C}\left\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\right\}} = C_{M_z} \tag{117}$$

Equations (112) through (117) can also be cast into matrix format for direct computer computation as follows:

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} \begin{bmatrix} C_{F_X} \\ C_{F_Y} \\ C_{F_Z} \end{bmatrix} = \begin{bmatrix} \dfrac{2\dot{\gamma}^2 RM}{\rho S\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\}} \\ 0 \\ -\dfrac{2Mg}{\rho S\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\}} \end{bmatrix} \tag{118}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{M_X} \\ C_{M_Y} \\ C_{M_Z} \end{bmatrix} = \begin{bmatrix} -\dfrac{2(I_X \sin^2\phi + I_Y \cos^2\phi - I_Z)\dot{\gamma}^2 \sin\theta\cos\theta}{\rho S\bar{c}\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\}} \\ 0 \\ -\dfrac{2(I_X - I_Y)\dot{\gamma}^2 \sin^2\theta \sin\phi\cos\phi}{\rho S\bar{c}\{(\dot{\gamma}R)^2+(\dot{z}_0)^2\}} \end{bmatrix} \tag{119}$$

where the fact that $C_{M_{Z_1}} = 0$, as expressed by equation (116), has been used in equation (114) to show that $C_{F_{Y_1}} = 0$ in the steady spin.

In addition to allowing the computation of the force and moment coefficients from steady spin parameters the results of this development show that C_{M_X}, C_{M_Y} and C_{M_Z} are not

necessarily zero in the steady spin case, contrary to what one might have intuitively assumed. In fact, the steady spin situation where C_{M_X}, C_{M_Y}, and C_{M_Z} are all zero is a special case not likely to be encountered in airplanes of conventional design. To illustrate this point, equations (115) and (117) can be solved for the case where the moment coefficients are all zero with the result that:

$$I_X \sin^2\phi + I_Y \cos^2\phi = I_Z \qquad (120)$$

and

$$I_X = I_Y \qquad (121)$$

or simply:

$$I_X = I_Y = I_Z \qquad (122)$$

Thus it is seen that in a conventional airplane where it is unrealistic to have equation (122) satisfied, aerodynamic moments must be present to balance the gyroscopic moments and gyroscopically precess the inertial orientation of the airplane sufficiently to maintain a fixed orientation relative to the spin axis and radius vector.

VII. RESULTS AND DISCUSSION

A. ANALYTICAL ASPECTS

Although the primary intent of this effort was to develop another method of computing airplane spin modes from wind tunnel test data, the equations of motion derived in Section IV can also be used to advantage in a qualitative analytical approach to the problem of determining spin characteristics. In addition to possibly suggesting new parametric indices of spin characteristics, these equations should also allow a number of experimentally observed effects of parameter variations to be put on a firmer analytical foundation, such as the effect of the sign of $(I_x - I_y)$ on spin recovery [Ref. 8]. The fact that the inertia characteristics appear so prominently in equations (79) through (84) confirms test results which have suggested that the gyroscopic aspects of the problem are as important as the aerodynamics. Another result of this analysis of the spin problem was to emphasize the non-linearity of the aerodynamics associated with the spin. Thus computational techniques which use superposition in modeling the aerodynamics or employ partially linearized equations of motion should be expected to yield rather poor results where high spin rates or small spin radii are involved. These are probably the main reasons why the conventional approach has not really been successful in treating spins in general,

and has been particularly unproductive in predicting flat spin modes.

B. THE COMPUTER PROGRAM

Although the suggested computer program outlined in Section V C uses conventional aerodynamic modeling which is not entirely representative in the spin, it is felt that development of a computer program capable of solving the equations of motion is the next essential step in this approach to the spin problem. Once the computer program is working, improvements in the modeling of the forces and mements can be incorporated as they are devised. Thus the computational and aerodynamic parts of the problem can be pursued, in the interim, as separate problems. It is with this in mind that the author has chosen to discuss a few of the aspects of the computer solution problem rather than address the aerodynamics of the problem.

The iterative scheme outlined in Section V B may exhibit a tendency toward a lightly damped oscillatory form of convergence since it is essentially following the dynamical motion of the airplane, which is characteristically lightly damped in a spin. The oscillatory nature of the convergence is compounded somewhat because we cannot be sure how far a starting guess may be from a force and moment equilibrium. To improve this situation there are a number of ways to provide some synthetic damping to the iteration to speed convergence. Some of these ways are:

a. If excursions in radius prove to be a problem as a result of momentum in radius rate ($\dot{R}$), the motion can be damped as follows;

$$\dot{R}_{(n+1)} = \zeta \dot{R}_{(n)} + \ddot{R}\Delta t \tag{123}$$

where ζ is some factor between zero and one.

b. If excursions in altitude rate become a problem they can be handled as follows;

$$\dot{z}_{(n+1)} = \dot{z}_{(n)} + \zeta \ddot{z}\Delta t \tag{127}$$

Another problem that may arise is that convergence in orientation may be deadbeat, depending on the size of Δt. This is due to the fact that we have eliminated orientation rates to reduce any tumbling tendencies. Thus it may be necessary to augment the rate of change in orientation per iteration to speed convergence. This can be done in the following manner:

$$\theta_{(n+1)} = \theta_{(n)} + b\left(\frac{\ddot{\theta}\Delta t^2}{2}\right) \tag{125}$$

$$\psi_{(n+1)} = \psi_{(n)} + b\left(\frac{\ddot{\psi}\Delta t^2}{2}\right) \tag{126}$$

$$\phi_{(n+1)} = \psi_{(n)} + b\left(\frac{\ddot{\phi}\Delta t^2}{2}\right) \tag{127}$$

Where b is some factor greater than one.

Finally, since our starting guess in $\dot{\gamma}$ may be out of the ball park and cause large changes in radius prior to reaching

a magnitude consistent with the aerodynamics of the problem, it may be necessary to augment the rate of change of this characteristically unresponsive parameter. This may be accomplished in the following way:

$$\dot{\gamma}_{(n+1)} = \dot{\gamma}_{(n)} + C\left(\ddot{\gamma}\Delta t\right) \tag{128}$$

where C is some factor greater than one.

C. FOLLOW-ON STUDIES

As mentioned earlier, the equations of motion as derived in this paper should make it possible to provide an analytical basis for many of the heretofore experimentally observed characteristics of airplane spinning motion. Such a correlation study would be relatively inexpensive to pursue and would probably provide a wealth of additional insight into the spin problem as a whole and possibly be beneficial to the more immediate goals of computer program development and force-moment modeling. It might also be instructive to program the full equations of motion, as contained in the appendix, on an analog computer having a digital computer interface capability for the aerodynamic data. Some other aspects of the spin problem which could be pursued once the computer programming and force modeling problems are solved are:

(1) Determine the spin modes of currently operated tactical military airplanes and investigate the effectiveness of recommended spin recovery techniques.

(2) Determine what characteristic aerodynamic parameters can be most economically fixed in existing airplanes with irrecoverable spins and investigate the magnitudes of such changes that would be required to make recovery possible. An example would be providing rudder stop over-rides or additional deflection for spin recoveries.

(3) Study the feasibility of providing strap-on devices such as a deployable fabric ventral fins or wing mounted retro-rocket pods for use on tactics training flights. The size of the device or necessary thrust requirements could be fairly easily determined with the assistance of a working computer program, or, in the case of the retro-rocket, could also be determined by the "Inverse Problem" Technique outlined in Section VI.

(4) Determine "design envelopes" that prescribe limits in terms of common aerodynamic derivatives in order to assist the designer in avoiding troublesome combinations wherever feasible. Such work would also provide the basis for improvements in the current Military Specification MIL F-8785 (ASG).

(5) Study the feasibility of utilizing an autopilot to apply oscillatory control inputs in resonance with the dynamics of the spin to break an otherwise irrecoverable mode.

These are only a few applications that might be undertaken to alleviate the spin problems inherent in the designs of todays high performance tactical military

airplanes. However the satisfactory accomplishment of just a few is all that is really needed.

VIII. CONCLUSIONS AND RECOMMENDATIONS

The approach to the spin problem outlined in this paper should provide a means of obtaining some additional insight into the problem of aircraft spin. Once a successful computer program has been written and a more representative means of modeling aerodynamic forces and moments has been devised it should be possible to solve the equations of motion and obtain some idea of the spin modes that a given airplane design may exhibit. The results of Section VI indicate that gyroscopic moments play an important role in the spin problem. Thus it may also be worthwhile to treat the spinning airplane as a slow turning free gyroscope, of relatively high inertia, being acted upon by triaxial aerodynamic forces and moments. Such a formulation of the spin would allow the application of the broad scope of knowledge of gyroscopic motion from the fields of physics and mechanics to the airplane spin problem. In this manner it may be possible to obtain some additional understanding of the stability of various spin modes and possibly avoid some undesirable airplane spin characteristics without having detailed knowledge of the aerodynamics involved.

APPENDIX

SPIN EQUATIONS

The following equations constitute the full equations of motion for a spinning airplane as written in a cylindrical coordinate system.

Z Equation

$$M\ddot{Z}_0 + Mg = -(C_{F_X}\alpha_{13} + C_{F_Y}\alpha_{23} + C_{F_Z}\alpha_{33})\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\}$$

R Equation

$$M\ddot{R} - M\dot{\gamma}^2 R = -(C_{F_X}\alpha_{11} + C_{F_Y}\alpha_{21} + C_{F_Z}\alpha_{31})\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\}$$

γ Equation

$$MR(2\dot{R}\dot{\gamma} + R\ddot{\gamma}) + I_x\left\{(\ddot{\psi} + \ddot{\gamma})\alpha_{13} + (\dot{\psi} + \dot{\gamma})(\dot{\theta}\cos\theta\sin\phi\right.$$

$$\left. + \dot{\phi}\sin\theta\cos\phi) + \ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi\right\}\alpha_{13} + I_x\left\{(\dot{\psi} + \dot{\gamma})\alpha_{13}\right.$$

$$\left. + \dot{\theta}\cos\phi\right\}\left\{\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right\} + I_y\left\{(\ddot{\psi} + \ddot{\gamma})\alpha_{23}\right.$$

$$+ (\dot{\psi} + \dot{\gamma})(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) - \ddot{\theta}\sin\phi$$

$$\left. - \dot{\theta}\dot{\phi}\cos\phi\right\}\alpha_{23} + I_y\left\{(\dot{\psi} + \dot{\gamma})\alpha_{23} - \dot{\theta}\sin\phi\right\}\left\{\dot{\theta}\cos\theta\cos\phi\right.$$

$$\left. - \dot{\phi}\sin\theta\sin\phi\right\} + I_z\left\{(\ddot{\psi} + \ddot{\gamma})\alpha_{33} + (\dot{\psi} + \dot{\gamma})(-\dot{\theta}\sin\theta)\right.$$

$$\left. + \ddot{\phi}\right\}\alpha_{33} + I_z\left\{(\dot{\psi} + \dot{\gamma})\alpha_{33} + \dot{\phi}\right\}\left\{-\dot{\theta}\sin\theta\right\}$$

$$= \left\{\bar{c}(C_{M_X}\alpha_{13} + C_{M_Y}\alpha_{23} + C_{M_Z}\alpha_{33}) - R(C_{F_X}\alpha_{12} + C_{F_Y}\alpha_{22}\right.$$

$$\left. + C_{F_Z}\alpha_{32})\right\}\frac{\rho S}{2}\left\{(\dot{\gamma}R)^2 + (\dot{R})^2 + (\dot{Z}_0)^2\right\}$$

θ Equation

$$I_X\Big\{(\ddot{\psi}+\ddot{\gamma})\alpha_{13} + (\dot{\psi}+\dot{\gamma})(\dot{\theta}\cos\theta\sin\phi+\dot{\phi}\sin\theta\cos\phi)$$
$$+\ddot{\theta}\cos\phi-\dot{\theta}\dot{\phi}\sin\phi\Big\}\cos\phi-I_X\dot{\phi}\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{13}+\dot{\theta}\cos\phi\Big\}\sin\phi$$
$$-I_Y\Big\{(\ddot{\psi}+\ddot{\gamma})\alpha_{23}+(\dot{\psi}+\dot{\gamma})(\dot{\theta}\cos\theta\cos\phi-\dot{\phi}\sin\theta\sin\phi)$$
$$-\ddot{\theta}\sin\phi-\dot{\theta}\dot{\phi}\cos\phi\Big\}\sin\phi-I_Y\dot{\phi}\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{23}$$
$$-\dot{\theta}\sin\phi\Big\}\cos\phi-I_X\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{13}+\dot{\theta}\cos\phi\Big\}\Big\{(\dot{\psi}$$
$$+\dot{\gamma})\cos\theta\sin\phi\Big\}-I_Y\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{23}-\dot{\theta}\sin\phi\Big\}\Big\{(\dot{\psi}$$
$$+\dot{\gamma})\cos\theta\cos\phi\Big\}+I_Z\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{33}+\dot{\phi}\Big\}\Big\{(\dot{\psi}+\dot{\gamma})\sin\theta\Big\}$$
$$=(C_{MX}\cos\phi-C_{MY}\sin\phi)\frac{\rho S\bar{C}}{2}\Big\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_0)^2\Big\}$$

ψ Equation

$$I_X\Big\{(\ddot{\psi}+\ddot{\gamma})\alpha_{13}+(\dot{\psi}+\dot{\gamma})(\dot{\theta}\cos\theta\sin\phi+\dot{\phi}\sin\theta\cos\phi)+\ddot{\theta}\cos\phi$$
$$-\dot{\theta}\dot{\phi}\sin\phi\Big\}\alpha_{13}+I_X\Big\{(\dot{\psi}+\dot{\gamma})\alpha_{13}+\dot{\theta}\cos\phi\Big\}\Big\{\dot{\theta}\cos\theta\sin\phi$$
$$+\dot{\phi}\sin\theta\cos\phi\Big\}+I_Y\Big\{(\ddot{\psi}+\ddot{\gamma})\alpha_{23}+(\dot{\psi}+\dot{\gamma})(\dot{\theta}\cos\theta\cos\phi$$
$$-\dot{\phi}\sin\theta\sin\phi)-\ddot{\theta}\sin\phi-\dot{\theta}\dot{\phi}\cos\phi\Big\}\alpha_{23}+I_Y\Big\{(\dot{\psi}$$
$$+\dot{\gamma})\alpha_{23}-\dot{\theta}\sin\phi\Big\}\Big\{\dot{\theta}\cos\theta\cos\phi-\dot{\phi}\sin\theta\sin\phi\Big\}$$
$$+I_Z\Big\{(\ddot{\psi}+\ddot{\gamma})\alpha_{33}+(\dot{\psi}+\dot{\gamma})(-\dot{\theta}\sin\theta)+\ddot{\phi}\Big\}\alpha_{33}+I_Z\Big\{(\dot{\psi}$$
$$+\dot{\gamma})\alpha_{33}+\dot{\phi}\Big\}\Big\{-\dot{\theta}\sin\theta\Big\}$$
$$=(C_{MX}\alpha_{13}+C_{MY}\alpha_{23}+C_{MZ}\alpha_{33})\frac{\rho S\bar{C}}{2}\Big\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_0)^2\Big\}$$

ϕ Equation

$$I_z\left\{(\ddot{\psi}+\ddot{\gamma})\alpha_{33}+(\dot{\psi}+\dot{\gamma})(-\dot{\theta}\sin\theta)+\ddot{\phi}\right\}-I_x\left\{(\dot{\psi}+\dot{\gamma})\alpha_{13}\right.$$

$$\left.+\dot{\theta}\cos\phi\right\}\left\{(\dot{\psi}+\dot{\gamma})\sin\theta\cos\phi-\dot{\theta}\sin\phi\right\}-I_y\left\{(\dot{\psi}+\dot{\gamma})\alpha_{23}\right.$$

$$\left.-\dot{\theta}\sin\phi\right\}\left\{-(\dot{\psi}+\dot{\gamma})\sin\theta\sin\phi-\dot{\theta}\cos\phi\right\}$$

$$=(C_{M_z})\frac{\rho S\bar{c}}{2}\left\{(\dot{\gamma}R)^2+(\dot{R})^2+(\dot{z}_0)^2\right\}$$

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13. ABSTRACT

The general equations of motion for a rigid body are derived in cylindrical coordinates by Lagrangian dynamics and used to model the motion of an airplane in a steady spin. After simplification, the equations are cast into a form utilizing conventional aerodynamic derivatives along with other derivatives which may be significant in spins. An iterative numerical solution procedure is outlined which should simplify the problem of solving the nonlinear differential equations, and relationships between the Euler Angles used in the equations and the more familiar ordered set of pitch, roll, and yaw are derived to permit computer input and output of orientation to be more easily visualized. The inverse problem is also considered, and equations are derived to translate spin test data into the six Lagrange coordinates used so that computer calculations may be checked against test results and actual aerodynamic forces can be compared with computed values. The paper concludes with a discussion of the implications of this spin analysis along with applications and extensions.

Security Classification

KEY WORDS	LINK A		LINK B		LINK C	
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